Effective Field Theory

- Tim Cohen's lectures on EFT (1903.03622)
  ch 1, 2, 3.

We are familiar with the idea that one does not need to know the detailed microscopic properties of a system to describe it.
This is true in classical physics, we don't need a molecular model to describe waves in the sea, and it is true in QM, as we can consider the proton point-like in the hydrogen atom.
In this case, $r_B \sim 10^{-10}\,m$, while $r_p \sim 10^{-15}\,m$, so there is a ratio of scales $r_p/r_B \sim 10^{-5}$.
An Effective Field Theory is a description

of a system where these type of approximations are systematically used and organized.

- Reminder of some terminology used.

  relevant coupling: positive mass dim ($m^2\phi^2$, $\mu\phi^3$)
  marginal coupling: zero mass dim ($\lambda\phi^4$)
  irrelevant coupling: negative mass dim ($\frac{1}{M^2}\phi^6$)

  The names refer to the fact that we will be interested in the low energy, and the typical strength of each operator is controlled by their mass dimension.

- Example: two real scalar fields.

  Take real fields ϕ & η, with $m_\phi \ll m_\eta$.

  Assume a Z_2: $\phi \to -\phi$, $\eta \to \eta$.

Then the most general Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m_\phi^2\phi^2 + \frac{1}{2}(\partial\eta)^2 - \frac{1}{2}m_\eta^2\eta^2 - V$$

$$; \; V = \frac{\lambda}{4!}\phi^4 + \frac{g}{2}\phi^2\eta + \frac{g'}{3!}\eta^3 + \frac{\lambda'}{4}\phi^2\eta^2 + \frac{\lambda''}{4!}\eta^4$$

The path integral to compute ϕ corr. functions is given by

$$Z[J] = \int D\phi D\eta \, e^{iS[\phi,\eta] + \int J\phi}$$

We include a source for ϕ, but not for η. This is because η is assumed to be heavy enough so that it only affects processes by virtual effects.

We can define the EFT path integral

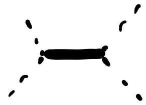
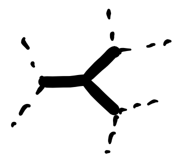
$$Z_{eff}[J] = \int D\phi \, e^{iS_{eff}[\phi] + \int J\phi}$$

so that

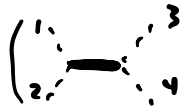
$$e^{iS_{eff}[\phi]} = \int D\eta \, e^{iS[\phi,\eta]}$$

This is useful if S_{eff} is local, meaning that $\mathcal{L}_{eff}$ is a polynomial in fields and in derivatives of fields.

The procedure to compute S_{eff} is to "integrate out" the heavy field η. Can be done perturbatively,

$$i S_{eff}[\phi] = i S[\phi] +$$  $$+$$  $$+ \dots$$

- Focus on the 4-pt:

$$=$$  $$+ \Big($$ 

$$+ \, t + u \Big)$$

$$= -i\lambda - i g^2 \left[\frac{1}{s - m_\eta^2} + \frac{1}{t - m_\eta^2} + \frac{1}{u - m_\eta^2} \right]$$

Assuming $E \ll m_\eta$,

$$\frac{1}{p^2 - m_\eta^2 + i\varepsilon} = -\frac{1}{m_\eta^2} - \frac{p^2}{m_\eta^4} + \dots$$

$\Rightarrow$ i  $= \lambda - 3\dfrac{g^2}{m_\eta^2} - \dfrac{g^2}{m_\eta^4}(s+t+u)$

$$- \frac{g^2}{m_\eta^6}(s^2+t^2+u^2) + \cdots$$

$$\Rightarrow \mathcal{L}_{eff}^{(4)} = -\frac{1}{4!}\left(\lambda - \frac{3g^2}{m_\eta^2}\right)\phi^4 - \frac{g^2}{8m_\eta^4}\phi^2\Box\phi^2$$

$$- \frac{g^2}{4!\,m_\eta^4}(\partial\phi)^4 + \cdots$$

So at low energies, $E/m_\eta \ll 1$, the η propagator is substituted by local derivative interactions of the low energy fields ϕ.

Formally, by writing

$$\frac{1}{p^2+M^2} = \frac{1}{M^2}\frac{1}{1+p^2/M^2} = \frac{1}{M^2}\sum_{n=0}^{\infty}\left(\frac{-p^2}{M^2}\right)^n$$

the (Euclidean) Feynman propagator is

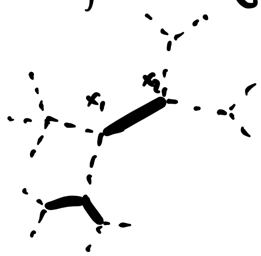
$$\Delta_F(x-y) = \int\frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-y)}\frac{1}{p^2+M^2}$$

$$= \frac{1}{M^2}\sum_{n=0}^{\infty} \int \frac{d^4p}{(2\pi)^4}\left(\frac{-p^2}{M^2}\right)^n e^{-ip(x-y)}$$

$$= \frac{1}{M^2}\sum_{n=0}^{\infty} \frac{\Box^n}{M^{2n}} \underbrace{\int\frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-y)}}_{\delta^4(x-y)}$$

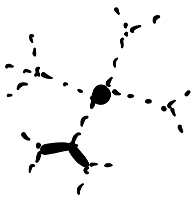
$$= \frac{1}{M^2}\sum_{n=0}^{\infty} \frac{\Box^n}{M^{2n}}\,\delta^4(x-y)$$

So for any diagram of the type



$$\sim \int dx_1\, dx_2\, J_1(x_1)\,\Delta_F(x_1-x_2)\,J_2(x_2)$$

$$= \int dx\, \frac{1}{M^2}\sum_n J_1(x)\left(\frac{\Box}{M^2}\right)^n J_2(x)$$



So the effect of the propagator is to generate a tower of operators in the low energy theory.

This is only useful if we can truncate the sum. This can be done if the expression above is convoluted with slowly varying functions, $\frac{\partial}{\partial x} f \ll M f$.

This is the case of scattering amplitudes at low energy, since it contains $e^{ip\cdot x}$.

- Integrate out heavy field.

We can integrate out η by using the equations of motion.

$$S_{eff}[\phi] = S[\phi, \eta_{cl}] + \underset{(\text{loops})}{\mathcal{O}(\hbar)}$$

with $$\left.\frac{\delta S[\phi,\eta]}{\delta \eta}\right|_{\eta=\eta_{cl}} = 0.$$

EOM: $$\Box \eta + m_\eta^2 \eta + \frac{g}{2}\phi^2 + \frac{g'}{2}\eta^2 + \frac{\lambda'}{2}\eta\phi^2 + \frac{\lambda''}{6}\eta^3 = 0$$

We can solve iteratively.

Take the power counting

$$m_\eta \sim M, \quad g \sim g' \sim M, \quad \lambda\text{'s} \sim 1.$$

Then, $\eta_{cl}^{(1)} \simeq -\frac{g}{2m_\eta^2}\phi^2 \sim \mathcal{O}(\frac{1}{M})$

$$\eta_{cl} = \underbrace{-\frac{g}{2m_\eta^2}\phi^2}_{1/M} \underbrace{- \frac{1}{m_\eta^2}\Box\eta_{cl}}_{1/M^3} \underbrace{- \frac{\lambda'}{2m_\eta^2}\eta_{cl}\phi^2}_{1/M^3} \underbrace{- \frac{g'}{2m_\eta^2}\eta_{cl}^2}_{1/M^3} \underbrace{- \frac{\lambda''}{6m_\eta^2}\eta_{cl}^3}_{1/M^5}$$

To go to order M^3 we substitute $\eta_{cl}^{(1)}$ to the EOM, keeping terms up to $1/M^3$.

$$\Rightarrow \eta_{cl}^{(3)} = -\frac{g}{2m_\eta^2}\phi^2 + \frac{g}{2m_\eta^4}\Box\phi^2 + \left(\frac{g\lambda'}{4m_\eta^4} + \frac{g^2 g'}{4m_\eta^6}\right)\phi^4 + \mathcal{O}\left(\frac{1}{M^5}\right)$$

Substituting into $\mathcal{L}_{UV}$,

$$\begin{aligned} \mathcal{L}_{eff} = & \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m_\phi^2\phi^2 - \frac{1}{4!}\left(\lambda - \frac{3g^2}{m_\eta^2}\right)\phi^4 \\ & - \frac{1}{6!}\left(\frac{45\lambda' g^2}{m_\eta^4} - \frac{15g'g^3}{m_\eta^6}\right)\phi^6 \\ & + \frac{g^2}{8m_\eta^4}(\partial_\mu\phi^2)(\partial^\mu\phi^2) + \mathcal{O}\left(\frac{1}{M^4}\right) \end{aligned}$$

Which agrees with the previous diagrammatic approach.

- Simplifying $\mathcal{L}_{eff}$

Lagrangians, in particular when many derivative interactions are involved, can be re-arranged & simplified by

  - integration by parts
  - field redefinitions

- Ex: it is possible to classify all possible terms of the form $\partial^2\phi^n$. Using $\partial\phi^r = r\phi^{r-1}\partial\phi$, we can write op so that ∂'s act on a single field.

The most general op is a linear combination of $\phi^{n-1}\Box\phi$ and $\phi^{n-2}\partial_\mu\phi\,\partial^\mu\phi$.

Using

$$\phi^{n-2}\partial_\mu\phi\,\partial^\mu\phi = \frac{1}{n-1}\partial^\mu\phi^{n-1}\partial_\mu\phi$$

$$= -\frac{1}{n-1}\phi^{n-1}\Box\phi + \text{total der.}$$

Therefore, there is a single indep.

operator at a given n.

- Ex: field redefinitions.

Do $\phi \to \phi + f(\phi)$ and expand in powers of f.

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - V$$

$$\downarrow$$

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - V - f(\phi)\underbrace{(\Box\phi + V')}_{\text{EOM}} + \mathcal{O}(f^2)$$

Let's simplify our previous example.

Do
$$\phi \to \phi + c\,\frac{g^2}{m_\eta^4}\phi^3$$

$$\Rightarrow \mathcal{L}_{eff} \to \mathcal{L}_{eff} + \frac{c g^2}{m_\eta^4}\phi^3\left[\Box\phi + m_\phi^2\phi + \frac{\lambda}{3!}\phi^3\right] + \ldots$$

Taking $c = 1/2$, we obtain

$$\mathcal{L}_{eff} \to \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m_\phi^2\phi^2$$

$$- \frac{1}{4!}\left(\lambda - \frac{3g^2}{m_\eta^2} - \frac{6g^2 m_\phi^2}{m_\eta^4}\right)\phi^4$$

$$+ \frac{1}{6!}\left[\frac{g^2(45\lambda' - 60\lambda)}{m_\eta^4} - \frac{15 g' g^3}{m_\eta^6}\right]\phi^6 + \mathcal{O}\left(\frac{1}{M^4}\right)$$

We have eliminated the $\partial^2\phi^4$ term!

All indirect effects from η captured by ϕ^4, ϕ^6 terms up to $\mathcal{O}(E^2/M^2)$

That $\partial^2\phi^4$ can be eliminated by field redefinition it means that leaves no imprint on physical observables.

This could be seen in the amplitude. It lead to a term

$$\propto s + t + u$$

But we have that

$$s = (p_1+p_2)^2 = 2p_1\cdot p_2 + 2m_\phi^2$$
$$t = (p_1-p_3)^2 = -2p_1\cdot p_3 + 2m_\phi^2$$
$$u = (p_1-p_4)^2 = -2p_1\cdot p_4 + 2m_\phi^2$$

$$\Rightarrow \quad s+t+u = 2p_1\cdot(p_2-p_3-p_4) + 6m_\phi^2$$
$$= -2p_1\cdot p_1 + 6m_\phi^2$$
$$= 4m_\phi^2$$

So the derivative structure disappears by momentum conservation, and the term is indeed equivalent to a redefinition of ϕ^4.

- Universality

Different UV theories lead to the same IR theory.

Example: take N fields η_i.

$$V = \frac{\lambda}{4!}\phi^4 + \frac{g_i}{2}\phi^2\eta_i + \frac{g'_{ijk}}{3!}\eta_i\eta_j\eta_k + \frac{\lambda'_{ij}}{4}\phi^2\eta_i\eta_j + \frac{\lambda''_{ijkl}}{4!}\eta_i\eta_j\eta_k\eta_l$$

Using the counting

$$m_{\eta_i} \sim g_i \sim M, \qquad \lambda\text{'s} \sim 1.$$

One gets

$$\mathcal{L}_{eff} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m_\phi^2\phi^2 - \frac{\lambda_{4,eff}}{4!}\phi^4 - \frac{\lambda_{6,eff}}{6!M^2}\phi^6$$

$$+ \ldots$$

with $\lambda_{4,eff} = \lambda - \sum_i \frac{3g_i^2}{m_{\chi_i}^2}$, etc.

• Bottom-up EFT

The above suggests another use of EFT's: to characterize the relevant effects at low energies.

We can just write down all operators allowed by symmetries. Higher dimensional operators will be naturally supressed if the scale that generates it is sufficiently high with respect the experimental energies.

- Masses, scales, and couplings.

So far we only kept track of mass dimensions.

This obscures an important aspect of power counting. Let's recover momentarily the $\hbar$ dimensions.

The action S has units of $\hbar$, as it appears as $e^{\frac{i}{\hbar}S}$ in the path integral.

So the Lagrangian has dim

$$[\mathcal{L}] = [\hbar]/L^4 = E\cdot T/L^4$$

So the canonically normalized bosonic & fermionic fields have dimension

$$[\phi] = [\hbar]^{1/2}/L \quad , \quad [\psi] = [\hbar]^{1/2}/L^{3/2}$$

Mass dimensions correspond to powers of L^{-1}.

Instead, the "coupling dimensions" are given by the powers of $[\hbar]^{-1/2}$ of a given parameter.

For instance,

$\lambda \phi^4 \rightarrow \lambda$ is dimensionless in mass,

but has dimensions of

coupling squared.

or,

$e \bar{\psi} \not{A} \psi \rightarrow e$ has dim. of coupling.

The 2-to-2 amplitude has dimensions of coupling squared, and that is why it is proportional to $\propto e^2$ & to $\propto \lambda$.

At loop level, extra powers of $\hbar$ leads to exta couplings.

This leads to an automatic power counting of the generated operators.

Take m_* to be the only UV scale, and g_* to be the only UV coupling.

Then, the generated Lagrangian

takes the form

$$\mathcal{L}_{EFT} = \frac{m_*^4}{g_*^2} \mathcal{L}_{tree}\left[\frac{\partial}{m_*}, \frac{g_* \phi}{m_*}\right]$$

In complicated theories, this counting may allow to understand which operators are relevant for weakly ($g_* \ll 1$) and strongly ($g_* \gtrsim 1$) coupled theories.

- **SM as an EFT & neutrino masses**

Now we understand why we've been ignoring operators of dim > 4. The SM is the low energy description, valid for energies $\ll m_*$, with m_* being the characteristic scale of the (unknown) dynamics out of which the SM emerges.
Therefore operators of dim ≤ 4 dominate

the low energy description.
Considering higher dimensional operators is equivalent to probe the leading effects of the microscopic theory.

The leading class of operators are those of dimension 5. There is a single structure, given by

$$\mathcal{L}_5 = \frac{c_5}{\Lambda}(H\bar{\ell}_L^c)(H\ell_L)$$

Note that each term $H\ell_L$ & $H\bar{\ell}_L^c$ is a singlet under the SM group.
After EWSB, one gets

$$\mathcal{L}_5 = \frac{1}{2}\frac{c_5}{\Lambda}v^2\bar{\nu}_L^c\nu_L$$

so it is a mass term for the neutrino!
The Majorana mass term that we had

to neglect last lecture, now it is generated by a dim 5 operator after the Higgs takes a vev.

So the mechanism to generate ν masses is similar to the other fermions, but with the crucial difference that it originates from a dim 5 op.

Given that

$$m_\nu \sim \frac{c_5 v}{\Lambda} v$$

If $m_\nu \sim 0.1\text{eV}$ (hasn't been measured though) then for $c_5 \sim 1$ one gets

$$\Lambda \sim 10^2\,\text{GeV}\,\frac{10^2}{0.1\cdot 10^{-9}} \sim 10^{14}\,\text{GeV}$$

The origin of the smallness of ν masses is understood: they are associated with an irrelevant operator generated at a very high scale.

- Accidental symmetries

We've been emphasizing the importance of the accidental symmetries of the SM. These appear because there are no operators of dim ≤ 4 that break them. In general, these are broken by microscopic dynamics, but the low energy theory will be sensitive to this breaking only at higher orders in the EFT expansion.

For instance,

$$\mathcal{L}_{\mu e \gamma} = \frac{c^B_{ij}}{\Lambda^2} \bar{\ell}^i_L \sigma_{\mu\nu} e^j_R H B^{\mu\nu} + \frac{c^W_{ij}}{\Lambda^2} \bar{\ell}^i_L \sigma_{\mu\nu} \tau^A H e^j_R W^A_{\mu\nu}$$

depend on a matrix of couplings c^B_{ij}

and c^w_{ij} that can be misaligned with the mass basis, and may lead to $\mu \to e\gamma$, $\mu \to eee$ processes.

Baryon number is also broken by

$$\mathcal{L}_{\not{B}} \supset \frac{c}{\Lambda^2} \bar{d}^c_R u_R \bar{q}^c_L l_L$$

which leads to proton decay.

So operators that are "irrelevant" in the RG-sense may become very relevant, in the phenomenological sense, if they are the first ones breaking a global symmetry.